A REDUCTION OF CERTAIN ANALYTIC DIFFERENTIAL EQUATIONS TO DIFFERENTIAL EQUATIONS OF AN ALGEBRAIC TYPE*

BY

WILLIA I DUNCAN MACMILLAN

Introduction

The theorem that a system of power series in n dependent variables and p independent variables can be reduced to an equivalent system which are polynomials with respect to the dependent variables† leads rather naturally to the suggestion that possibly a corresponding reduction can be effected in the case of analytic differential equations. It might be anticipated that the theory for such a reduction of differential equations is a more complicated one than the corresponding theory for implicit functions. In certain respects this anticipation is amply justified, while in certain other respects, perhaps, it is not.

In effecting such a reduction many different methods and points of view are possible. The method adopted in the present paper is a substitution on the dependent variables of the form

$$x_i = \sum_{j=1}^{n} a_{ij}\, y_j + \text{terms of higher degree},$$

where the determinant $|a_{ij}|$ is distinct from zero. Such a substitution might well be called a *linear-transcendental* substitution.‡ It is certainly transcendental in general, and it partakes of the nature of a linear substitution in that, for small values of the variables, the correspondence between the variables x and y is unique. We seek then to determine a substitution of this type such that the resulting differential equations in the y's are algebraic.

Let the differential equations be of the form

$$\frac{dx_i}{dt} = X_i(x_1, \cdots, x_n) \qquad (i = 1, \cdots, n),$$

* Presented to the Society, December 26, 1913, and April 10, 1914.

† MacMillan, Mathematische Annalen, LXXII (1912), pp. 157–179.

‡ Since this paper was written (1911) there has appeared a paper by H. Dulac on the integrals of differential equations in which he uses a transformation which turns out to be a linear-transcendental substitution. His methods, however, are quite different from those employed here. See H. Dulac, Bulletin de la Société Mathématique de France, vol. 40 (1912), pp. 324–383.

in which the X_i are expansible in powers of $x_1, \cdots, x_n$ and vanish with these variables. In the present paper it will be supposed that the determinant of the linear terms of the X_i is distinct from zero, and that its "equation in s," formed by subtracting s from each term of the main diagonal of the determinant, is such that all of its roots, considered as points in the complex plane, lie on one side of a straight line passing through the origin and at a finite distance from it. No hypothesis is made with respect to the nature of the multiplicity of the roots. Under these conditions it will be shown that, aside from certain exceptional cases which are fully discussed, a substitution exists such that the differential equations in the y's are identical with the linear terms of the differential equations in the x's, and in the exceptional cases they are simply related to the linear terms. If the hypothesis that *all* of the roots of the equation in s lie on one side of a straight line through the origin is not satisfied then, unless $s = 0$ is a root of multiplicity n, one can always draw a line through the origin which will leave certain of the roots at a finite distance on one side, let us say p such roots. Then there exists a substitution which involves only p of the variables y such that the differential equations for the y's are linear. The solution for the x's in this event will not be complete for they involve only $p < n$ arbitrary constants.

Case I

1. Roots of the Equation in s are all Simple

Let us suppose that the differential equations are

$$\frac{d\xi_i}{dt} = \Xi_i(\xi_1, \cdots, \xi_n) \qquad (i = 1, \cdots, n), \tag{1}$$

where the Ξ_i are analytic functions of $\xi_1, \cdots, \xi_n$ which do not contain t. Let us suppose that these equations are satisfied by the constant solutions $\xi_i = \xi_i^{(0)}$. We wish to examine the solutions of (1) in the neighborhood of $\xi_i = \xi_i^{(0)}$. Accordingly we write $\xi_i = \xi_i^{(0)} + \xi_i^{(1)}$ and the differential equations become

$$\frac{d\xi_i^{(1)}}{dt} = A_{i1}\,\xi_1^{(1)} + \cdots + A_{in}\,\xi_n^{(1)} + \text{terms of higher degree,} \tag{2}$$

where the A_{ij} are constants.

Suppose now that the roots of the equation in s,

$$\begin{vmatrix} A_{11} - s, & A_{12}, & \cdots, & A_{1n} \\ A_{21}, & A_{22} - s, & \cdots, & A_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ A_{n1}, & A_{n2}, & \cdots, & A_{nn} - s \end{vmatrix} = 0, \tag{3}$$

are all simple and distinct from zero. Let these roots be $\sigma_1, \cdots, \sigma_n$. Then, according to the theory of linear substitutions, there exists always a linear substitution

$$\xi_i^{(1)} = a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n,$$

the determinant of which is not zero, which transforms equations (2) into the canonical form

$$\frac{dx_i}{dt} = \sigma_i x_i + X_i \qquad (i = 1, \cdots, n), \tag{4}$$

where the X_i are power series in $x_1, \cdots, x_n$ of order not less than two; that is, the X_i do not contain any terms of degree less than two in $x_1, \cdots, x_n$.

We seek now a substitution

$$x_i = y_i + P_i(y_1, \cdots, y_n), \tag{5}$$

where the P_i are power series in $y_1, \cdots, y_n$ of order not less than two, say

$$P_i = \sum \alpha_{j_1 \cdots j_n}^{(i)} y_1^{j_1} \cdots y_n^{j_n} \quad (j_1 + j_2 + \cdots + j_n \geqq 2), \tag{6}$$

which will simplify equations (4) as much as possible. The substitution of (5) in (4) gives rise to the equations

$$\begin{aligned}
&\left[1 + \frac{\partial P_1}{\partial y_1}\right]\frac{dy_1}{dt} + \frac{\partial P_1}{\partial y_2}\frac{dy_2}{dt} + \cdots + \frac{\partial P_1}{\partial y_n}\frac{dy_n}{dt} - \sigma_1 y_1 - \sigma_1 P_1 \\
&\qquad\qquad\qquad = Y_1(y_j + P_j), \\
&\frac{\partial P_2}{\partial y_1}\frac{dy_1}{dt} + \left[1 + \frac{\partial P_2}{\partial y_2}\right]\frac{dy_2}{dt} + \cdots + \frac{\partial P_2}{\partial y_n}\frac{dy_n}{dt} - \sigma_2 y_2 - \sigma_2 P_2 \\
&\qquad\qquad\qquad = Y_2(y_j + P_j), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\frac{\partial P_n}{\partial y_1}\frac{dy_1}{dt} + \frac{\partial P_n}{\partial y_2}\frac{dy_2}{dt} + \cdots + \left[1 + \frac{\partial P_n}{\partial y_n}\right]\frac{dy_n}{dt} - \sigma_n y_n - \sigma_n P_n \\
&\qquad\qquad\qquad = Y_n(y_j + P_j).
\end{aligned} \tag{7}$$

Since these equations are linear in the dy_i/dt they can be solved for these quantities, but it is simpler to assume a solution for the dy_i/dt and then to examine whether equations (7) can be satisfied. Accordingly, we seek to determine the P_i so that the y_i shall satisfy the equations

$$\frac{dy_i}{dt} = \sigma_i y_i, \tag{8}$$

which are the same as the linear terms of (4).

With these values of the dy_i/dt equations (7) become

$$\sum_{j=1}^{n} \sigma_j y_j \frac{\partial P_i}{\partial y_j} - \sigma_i P_i = Y_i \qquad (i = 1, \cdots, n), \tag{9}$$

where the Y_i are the results of substituting (5) in the X_i. They are power series in $y_k + P_k$ $(k = 1, \cdots, n)$ of order not less than two in these quantities. Since the P_i are of order two in the y_k equations (9) can be satisfied. For let us substitute (6) in (9) and see whether the coefficients of the substitution, $\alpha^{(i)}_{j_1 \ldots j_n}$, can be determined. If we denote the expanded form of the right members by

$$Y_i = \sum \beta^{(i)}_{j_1 \ldots j_n} y_1^{j_1} \cdots y_n^{j_n},$$

we find, on comparing the right and left members of (9), that

$$[j_1 \sigma_1 + j_2 \sigma_2 + \cdots + j_n \sigma_n - \sigma_i]\, \alpha^{(i)}_{j_1 \ldots j_n} = \beta^{(i)}_{j_1 \ldots j_n}. \tag{10}$$

Let us denote $j_1 + j_2 + \cdots + j_n$ simply by j. Then for $j = 2$ the right members of (10) are known constants, and the quantities $\alpha^{(i)}_{j_1 \ldots j_n}$ are uniquely determined. When the $\alpha^{(i)}_{j_1 \ldots j_n}$ for $j = 2$ are known the $\beta^{(i)}_{j_1 \ldots j_n}$ for $j = 3$ are known; consequently the $\alpha^{(i)}_{j_1 \ldots j_n}$ for $j = 3$ are uniquely determined, and so on, sequentially. The process will fail if, and only if, for some set or sets of values $j_1, \cdots, j_n$ the coefficient

$$j_1 \sigma_1 + j_2 \sigma_2 + \cdots + j_n \sigma_n - \sigma_i = 0. \tag{11}$$

If the $\sigma_1, \cdots, \sigma_n$ are all real and of the same sign it can only happen exceptionally that (11) is satisfied. More generally, if the roots $\sigma_1, \cdots, \sigma_n$ considered as points in the complex plane lie all on one side of any straight line through the origin, it will be only exceptionally that expression (11) can be zero as was shown by Poincaré.*

Let us suppose that we are not dealing with one of the exceptional cases. Then we have formally determined a substitution (5) which transforms (4) into (8). Let us see whether this substitution converges. The solution of the equations

$$\begin{aligned} \sum (j_1 \sigma_1 + j_2 \sigma_2 + \cdots + j_n \sigma_n - \sigma_i)\, \alpha^{(i)}_{j_1 \ldots j_n} y_1^{j_1} \cdots y_n^{j_n} \\ = \sum \beta^{(i)}_{j_1 \ldots j_n} y_1^{j_1} \cdots y_n^{j_n} = Y_i(y_j + P_j) \\ (i = 1, \cdots, n), \end{aligned} \tag{12}$$

which are the same as (9), will be dominated by the solutions of the equations

$$\epsilon_i \sum \alpha^{(i)}_{j_1 \ldots j_n} y_1^{j_1} \cdots y_n^{j_n} = \bar{Y}_i(y_j + P_j) \qquad (i = 1, \cdots, n), \tag{13}$$

where the dash over the Y_i indicates that all of the coefficients in the expansion of the Y_i shall be given the positive sign, and ϵ_i is the smallest value of

$$|j_1 \sigma_1 + j_2 \sigma_2 + \cdots + j_n \sigma_n - \sigma_i|$$

for all $j_1, \cdots, j_n$. But equations (13) are the same as

* Thesis (1879).

$$\epsilon_i P_i = \bar{Y}_i(y_j + P_j). \tag{14}$$

Since the $\bar{Y}_i$ are of order not less than two in the arguments indicated, the solutions of (14) are convergent by the fundamental theorem of implicit functions, provided the $\bar{Y}_i(y_j + P_j)$ are convergent series. Hence the solutions of (9), which are the substitutions under discussion, are convergent.

This convergence proof, which is about as simple as a convergence proof can be, labors under the disadvantage that apparently it cannot be extended to more complicated cases. It will be useful therefore to give a second proof.

It will be no restriction of generality to suppose that the modulus of each coefficient in the expansions of the Y_i of equations (9) is less than $\nu_i > 0$, for if it were not true it would be easy to change the variables so that it would be true.

If the points σ_i in the complex plane are all at a finite distance from a straight line passing through the origin, and are all on one side of this straight line, then, if we are not dealing with one of the exceptional cases, there exists a positive number ϵ such that*

$$|j_1\sigma_1 + j_2\sigma_2 + \cdots + j_n\sigma_n - \sigma_i| > \epsilon|j_1 + j_2 + \cdots + j_n - 1|.$$

Consequently, if we take $\nu_i = \epsilon/n$ the solutions of (9) will be dominated by the solutions of

$$\epsilon\left[\sum_{k=1}^{n} y_k \frac{\partial Q_i}{\partial y_k} - Q_i\right] = \frac{\epsilon}{n} \frac{\left[\sum_{k=1}^{n}(y_k + Q_k)\right]^2}{1 - \sum_{k=1}^{n}(y_k + Q_k)}. \tag{15}$$

Since the Q_i are power series of order two it is clear that $Q_1 = Q_2 = \cdots = Q_n$. Taking then $y = \sum_{j=1}^{n} y_i$; $Q = \sum_{j=1}^{n} Q_j$, equations (15) reduce to

$$y\frac{dQ}{dy} - Q = \frac{(y+Q)^2}{1-(y+Q)}. \tag{16}$$

On integrating and choosing the constant of integration so that Q shall be of the second order in y, we get

$$Q = y(e^{(y+Q)} - 1), \tag{17}$$

or

$$Q = y^2 + \frac{3}{2}y^3 + \frac{8}{3}y^4 + \frac{125}{24}y^5 + \cdots.$$

Since the series (17) for Q as a power series in y are convergent, it follows that the $Q_i = Q/n$ are convergent, and consequently the solutions of (9) are convergent.

* See Picard: *Traité d'Analyse*, vol. III, p. 6.

It will be useful also to notice from (17) that

$$\frac{dQ}{dy} = \frac{e^{(y+Q)}}{1-(y+Q)} - 1$$

$$(18) \qquad = (y+Q) + \left(1+\frac{1}{2!}\right)(y+Q)^2 + \left(1+\frac{1}{2!}+\frac{1}{3!}\right)(y+Q)^3 + \cdots,$$

and that the coefficients of this expansion are all less than 3. Likewise, in the expansion (17),

$$(19) \qquad Q = y\left[(y+Q) + \frac{1}{2!}(y+Q)^2 + \frac{1}{3!}(y+Q)^3 + \cdots\right],$$

the coefficients of the right members are each less than or equal to unity.

We will lay aside now for a moment the hypothesis that it is possible to draw a straight line through the origin which will leave *all* of the points σ_i on one side of it. If the σ_i are all distinct and none of them zero, it will certainly be possible to draw a line which will leave at least one half of them on one side. Let us suppose that k of these roots $\sigma_1, \cdots, \sigma_k$ are thus situated, $k \geqq \frac{1}{2}n$.

In this event we can make, instead of (5), the substitution

$$(20) \qquad \begin{aligned} x_i &= y_i + P_i(y_1, \cdots, y_k) \qquad (i = 1, \cdots, k), \\ x_i &= 0 + P_i(y_1, \cdots, y_k) \qquad (i = k+1, \cdots, n). \end{aligned}$$

Then the equations corresponding to (7) are

$$(21) \qquad \begin{aligned} &\left[1+\frac{\partial P_1}{\partial y_1}\right]y_1' + \frac{\partial P_1}{\partial y_2}y_2' + \cdots + \frac{\partial P_1}{\partial y_k}y_k' - \sigma_1 y_1 - \sigma_1 P_1 = Y_1, \\ &\frac{\partial P_2}{\partial y_1}y_1' + \left[1+\frac{\partial P_2}{\partial y_2}\right]y_2' + \cdots + \frac{\partial P_2}{\partial y_k}y_k' - \sigma_2 y_2 - \sigma_2 P_2 = Y_2, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ &\frac{\partial P_k}{\partial y_1}y_1' + \frac{\partial P_k}{\partial y_2}y_2' + \cdots + \left[1+\frac{\partial P_k}{\partial y_k}\right]y_k' - \sigma_k y_k - \sigma_k P_k = Y_k, \\ &\frac{\partial P_{k+1}}{\partial y_1}y_1' + \frac{\partial P_{k+1}}{\partial y_2}y_2' + \cdots + \frac{\partial P_{k+1}}{\partial y_k}y_k' - 0 - \sigma_{k+1} P_{k+1} = Y_{k+1}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ &\frac{\partial P_n}{\partial y_1}y_1' + \frac{\partial P_n}{\partial y_2}y_2' + \cdots + \frac{\partial P_n}{\partial y_k}y_k' - 0 - \sigma_n P_n = Y_n, \end{aligned}$$

where

$$y_k' = \frac{dy_k}{dt}.$$

If now we take

$$P_i = \sum \alpha^{(i)}_{j_1 \cdots j_k} y_1^{j_1} \cdots y_k^{j_k} \qquad (j_1 + \cdots + j_k \geqq 2),$$

and

$$y_i' = \sigma_i y_i \qquad (i = 1, \cdots, k), \tag{22}$$

and if the coefficients in the expansions of the right members be denoted by $\beta^{(i)}_{j_1 \cdots j_k}$, we will obtain, on comparing coefficients,

$$[j_1 \sigma_1 + j_2 \sigma_2 + \cdots + j_k \sigma_k - \sigma_i]\, \alpha^{(i)}_{j_1 \cdots j_k} = \beta^{(i)}_{j_1 \cdots j_k}. \tag{23}$$

If then we exclude the exceptional cases, the $\alpha^{(i)}_{j_1 \cdots j_k}$ can be determined sequentially, just as before.

The solutions of equations (22), and therefore the expressions (20) for the x_i, contain only k arbitrary constants. The solutions thus obtained for equations (4) are incomplete, as we lack $n - k$ constants of integration.*

Case II

2. The Equation in s has Multiple Roots

When the equation in s has multiple roots the differential equations (2) can no longer be reduced to the simple form (4). They can always, however, be reduced to the form

$$\frac{dx_i}{dt} = \sum_{j=1}^{i-1} a_{ij} x_j + \sigma_i x_i + X_i \qquad (i = 1, \cdots, n), \tag{24}$$

whatever be the nature of the multiplicity of the roots of (3), where the constants a_{ij} are either zero or unity, and for a fixed i every a_{ij} is zero except perhaps one.

On making the substitution (5), viz.,

$$x_i = y_i + P_i(y_1, \cdots, y_n),$$

equations (24) become

* Since the solutions of equations (8) and (22) are $y_i = c_i e^{\sigma_i t}$, it follows that the x_i are expansible in powers of $c_i e^{\sigma_i t}$. This result was stated by Poincaré in his thesis for the case in which all of the σ_i lie on one side of a straight line through the origin and the condition $\Sigma j_k \sigma_k - \sigma_i \neq 0$ is satisfied. The case in which only k of the roots lie on one side of the straight line was discussed by Picard (*Traité D'Analyse*, vol. III). Thus the results obtained in Case I above are equivalent to those of Poincaré and Picard, though the method is different.

$$
(25)\qquad
\begin{aligned}
&\left[1+\frac{\partial P_1}{\partial y_1}\right]y_1'+\frac{\partial P_1}{\partial y_2}y_2'+\cdots+\frac{\partial P_1}{\partial y_n}y_n'-\sigma_1 y_1-\sigma_1 P_1 = Y_1,\\
&\frac{\partial P_2}{\partial y_1}y_1'+\left[1+\frac{\partial P_2}{\partial y_2}\right]y_2'+\cdots+\frac{\partial P_2}{\partial y_n}y_n'-\sigma_2 y_2-\sigma_2 P_2\\
&\qquad\qquad = Y_2+a_{21}(y_1+P_1),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\frac{\partial P_n}{\partial y_1}y_1'+\frac{\partial P_n}{\partial y_2}y_2'+\cdots+\left[1+\frac{\partial P_n}{\partial y_n}\right]y_n'-\sigma_n y_n-\sigma_n P_n\\
&\qquad\qquad = Y_n+\sum_{j=1}^{n-1}a_{nj}(y_j+P_j).
\end{aligned}
$$

If now we assume that (24) can be reduced to its linear terms, we will have

$$
(26)\qquad y_i' = \sum_{j=1}^{i-1} a_{ij}\,y_j+\sigma_i\,y_i,
$$

and the partial differential equations (25) defining the P_i are

$$
(27)\qquad
\begin{aligned}
&\sigma_1 y_1\frac{\partial P_1}{\partial y_1}+[a_{21}y_1+\sigma_2 y_2]\frac{\partial P_1}{\partial y_2}+\cdots+\left(\sum_{j=1}^{n-1}a_{nj}y_j+\sigma_n y_n\right)\frac{\partial P_1}{\partial y_n}\\
&\qquad\qquad -\sigma_1 P_1 = Y_1,\\
&\sigma_1 y_1\frac{\partial P_2}{\partial y_1}+[a_{21}y_1+\sigma_2 y_2]\frac{\partial P_2}{\partial y_2}+\cdots+\left(\sum_{j=1}^{n-1}a_{nj}y_j+\sigma_n y_n\right)\frac{\partial P_2}{\partial y_n}\\
&\qquad\qquad -\sigma_2 P_2 = Y_2+a_{21}P_1,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\sigma_1 y_1\frac{\partial P_n}{\partial y_1}+[a_{21}y_1+\sigma_2 y_2]\frac{\partial P_n}{\partial y_2}+\cdots+\left(\sum_{j=1}^{n-1}a_{nj}y_j+\sigma_n y_n\right)\frac{\partial P_n}{\partial y_n}\\
&\qquad\qquad -\sigma_n P_n = Y_n+\sum_{j=1}^{n-1}a_{nj}P_j.
\end{aligned}
$$

If now, bearing in mind the fact that the P_i are power series in $y_1,\cdots,y_n$ of order two, we write $P_i=\sum\alpha^{(i)}_{j_1\ldots j_n}y_1^{j_1}\cdots y_n^{j_n}$ and seek to determine the coefficients $\alpha^{(i)}_{j_1\ldots j_n}$ so as to satisfy (27), we find they are no longer determined by the simple formulæ (10), viz.,

$$
(j_1\sigma_1+j_2\sigma_2+\cdots+j_n\sigma_n-\sigma_i)\,\alpha^{(i)}_{j_1\ldots j_n}=\beta^{(i)}_{j_1\ldots j_n}.
$$

It is necessary to consider simultaneously the homogeneous terms (for each i) for which $j_1+j_2+\cdots+j_n=j$, $j=2,\cdots\infty$. This leads to systems of linear equations in the $\alpha^{(i)}_{j_1\ldots j_n}$ which if suitably arranged will have a determinant in which the coefficients $(j_1\sigma_1+j_2\sigma_2+\cdots+j_n\sigma_n-\sigma_i)$ occur in the main diagonal and all of the other elements of the determinant which are not zero lie on one side of the main diagonal. Hence the determinant is equal to

$$\Delta = \prod (j_1 \sigma_1 + j_2 \sigma_2 + \cdots + j_n \sigma_n - \sigma_i) \qquad (\Sigma j_k = j),$$

which can vanish if, and only if, for some set of values $j_1, \cdots, j_n$ we have

$$j_1 \sigma_1 + j_2 \sigma_2 + \cdots + j_n \sigma_n - \sigma_i = 0,$$

which is exactly the same condition (11) as in Case I.

Assuming that no such set of integers exists the coefficients $\alpha^{(i)}_{j_1 \dots j_n}$ can be determined uniquely so that equations (27) are satisfied. If the series thus obtained are convergent the differential equations (24) are reduced to their linear terms (26).*

The convergence proof is not so simple as in Case I, but the second proof given there can be modified so as to fit the present situation.

On rearranging the left members of (27), we have

$$(27') \quad \left[\sum_{k=1}^{n} \sigma_k y_k \frac{\partial P_i}{\partial y_k} - \sigma_i P_i \right] + \sum_{k=2}^{n} \left(\sum_{j=1}^{k-1} a_{kj} y_j \right) \frac{\partial P_i}{\partial y_k} - \sum_{j=1}^{i-1} a_{ij} P_j = Y_i \qquad (i = 1, \cdots, n).$$

The solutions of these equations are dominated by the solutions of the equations

$$(28) \qquad \epsilon \left[\sum_{k=1}^{n} y_k \frac{\partial Q_i}{\partial y_k} - Q_i \right] - \sum_{k=2}^{n} \left(\sum_{j=1}^{k-1} y_j \right) \frac{\partial Q_i}{\partial y_k} - \sum_{j=1}^{i-1} Q_j = \overline{Y}_i,$$

provided the $\overline{Y}_i$ dominate the Y_i. Now make the transformations

$$(29) \qquad y_i = \mu^{n-i} z_i, \qquad Q_i = \mu^{n-i} R_i, \qquad \overline{Y}_i = \mu^{n-i} Z_i.$$

After removing the factor μ^{n-i} equations (28) become

$$(30) \qquad \epsilon \left[\sum_{k=1}^{n} z_k \frac{\partial R_i}{\partial z_k} - R_i \right] - \sum_{k=2}^{n} \left(\sum_{j=1}^{k-1} \mu^{k-j} z_j \right) \frac{\partial R_i}{\partial z_k} - \sum_{j=1}^{i-1} \mu^{i-j} R_j = Z_i.$$

Obviously the solutions of (28) converge if the solutions of (30) converge.

For $\mu = 0$ equations (30) reduce to

$$(31) \qquad \epsilon \left[\sum_{k=1}^{n} z_k \frac{\partial R_i^{(0)}}{\partial z_k} - R_i^{(0)} \right] = Z_i^{(0)};$$

and if we take

$$Z_j^{(0)} = \frac{\epsilon}{n} \frac{[\sum (z_j + R_j^{(0)})]^2}{1 - \sum (z_j + R_j^{(0)})},$$

* The results obtained in Cases II and III are in agreement with those given by Dulac, *loc. cit.* though he did not give proofs of convergence.

equations (31) do not differ from (15) and therefore have the same solutions. From the properties of these solutions given in (18) and (19) it is easy to build a set of dominant functions Z_i for equations (30) for $\mu \neq 0$. Indeed, they will be so constructed that (30) and (31) have the same solutions. The equations which satisfy this condition are

$$\epsilon\left[\sum_{k=1}^{n} z_k \frac{\partial R_i}{\partial z_k} - R_i\right] - \sum_{k=2}^{n}\left(\sum_{j=1}^{k-1}\mu^{k-j} z_j\right)\frac{\partial R_i}{\partial z_k} - \sum_{j=1}^{i-1}\mu^{i-j} R_j$$

$$(32) \qquad = \frac{\epsilon}{n}\frac{[\sum (z_j + R_j)]^2}{1 - \sum(z_j + R_j)} - \frac{1}{n}\sum_{k=2}^{n}\left(\sum_{j=1}^{k-1}\mu^{k-j} z_j\right)\left(\frac{e^{\Sigma(z_j+R_j)}}{1 - \sum(z_j + R_j)} - 1\right)$$

$$- \frac{1}{n}\sum_{k=1}^{i-1}\mu^{i-k}\left(\sum_{j=1}^{n} z_j\right)\left(e^{\Sigma(z_j+R_j)} - 1\right).$$

If now we call $\sum_{k=1}^{n}\mu^k = m$, then the right members of these equations are expansible in powers of $\sum(z_j + R_j)$ with coefficients none of which are less than $(\epsilon - 4mn)/n$ which for sufficiently small values of μ is certainly positive, and the solutions of (32) are certainly convergent.

Returning now to equations (28) by means of (29), we see that the coefficients of Y_i are positive and not less than $(\epsilon - 4mn)/n\,\mu^{n-i}$. We have seen that without loss of generality the coefficients of the Y_i can be supposed less than $\nu_i > 0$. If then we take

$$\nu_i < \frac{\epsilon - 4mn}{n}\mu^n$$

the solutions of (30) will dominate the solutions of (28) which are accordingly convergent.

On combining the results of Cases I and II, we have the following theorem:

THEOREM. *If $dx_i/dt = X_i(x_1, \cdots, x_n)$ $(i = 1, \cdots, n)$ is a system of differential equations in which the X_i are analytic in $x_1, \cdots, x_n$, regular for $x_1 = \cdots = x_n = 0$ and vanish for these values of the variables; and if the equation in s, formed by subtracting s from the main diagonal of the functional determinant of the X_i for $x_1 = \cdots = x_n = 0$, has all of its roots $\sigma_1, \cdots, \sigma_n$ (considered as points in the complex plane) lying on one side of any straight line through the origin; and if there exists no set of integers $j_1, \cdots, j_n$ $(\sum_{k=1}^{n} j_k \geqq 2)$ such that $j_1\sigma_1 + j_2\sigma_2 + \cdots + j_n\sigma_n - \sigma_i = 0$, then there exists a linear-transcendental substitution*

$$x_i = \sum_{j=1}^{n}\alpha_{ij} y_i + \text{terms of higher degree}$$

such that the differential equations for the variables y are the same as the linear terms of the differential equations of the variables x, and such that the determinant $|\alpha_{ij}|$ is distinct from zero.

Case III

3. Exceptional Cases in which there Exist Sets of Integers $j_1, \cdots, j_n$, such that $\sum_{k=1}^{n} j_k \sigma_k - \sigma_i = 0$

We still retain the hypothesis that the points $\sigma_1, \cdots, \sigma_n$ lie on one side of a straight line through the origin. Notwithstanding this hypothesis it may happen that there exist sets of integers for which

$$j_1 \sigma_1 + j_2 \sigma_2 + \cdots + j_n \sigma_n - \sigma_i = 0.$$

From the preceding discussion it is clear that in this event it is not possible, in general, to reduce the differential equations to their linear terms. It is clear also that there can exist only a finite number of sets of integers for which the condition $\sum j_k \sigma_k - \sigma_i = 0$ is satisfied, for if we divide through by $\sum j_k = j$ we have

$$\frac{\sum j_k \sigma_k}{\sum j_k} = \frac{\sigma_i}{j}. \tag{33}$$

The left member is the expression for the center of gravity of masses equal to j_k situated at the points σ_k, and this center of gravity must lie within the polygon which encloses the points σ_k and cannot approach the origin. For increasingly large values of j the right member approaches zero. It is clear that (33) cannot be satisfied by any set of integers whose sum is sufficiently large.

Let us return now to equations (4), (5), and (6),

$$\frac{dx_i}{dt} = \sigma_i x_i + X_i, \tag{4}$$

$$x_i = y_i + P_i(y_1, \cdots, y_n), \tag{5}$$

$$P_i = \sum \alpha_{j_1 \cdots j_n}^{(i)} y_n^{j_1} \cdots y_n^{j_n} \qquad \left(\sum_{k=1}^{n} j_k \geqq 2\right). \tag{6}$$

Let us suppose that the maximum value of the sum, $j_1 + j_2 + \cdots + j_n$, for which $j_1 \sigma_1 + j_2 \sigma_2 + \cdots + j_n \sigma_n - \sigma_i = 0$ is $s - 1$.* Let us also assume the form of the differential equations for the y_i to be

$$\begin{aligned} \frac{dy_i}{dt} &= \sigma_i y_i + \sum{}^{*} b_{j_1 \cdots j_n}^{(i)} y_1^{j_1} \cdots y_n^{j_n} \qquad \left(\sum_{k=1}^{n} j_k \geqq 2\right), \\ &= \sigma_i y_i + p_i, \end{aligned} \tag{34}$$

where the terms included under the symbol $\sum^*$ are such that

$$j_1 \sigma_1 + j_2 \sigma_2 + \cdots + j_n \sigma_n - \sigma_i = 0.$$

* The s used here has no relation to the "equation in s" used previously.

If this expression for the y_i' be substituted in (7), we have for the determination of the P_i and the $b^{(i)}_{j_1 \cdots j_n}$ the equations

$$(35) \qquad \sum_{k=1}^{n} (\sigma_k y_k + p_k) \frac{\partial P_i}{\partial y_k} - \sigma_i P_i = Y_i - \sum{}^{*} b^{(i)}_{j_1 \cdots j_n} y_1^{j_1} \cdots y_n^{j_n}.$$

There is no difficulty in solving these equations, for when the coefficient of $\alpha^{(i)}_{j_1 \cdots j_n}$ vanishes the coefficient $b^{(i)}_{j_1 \cdots j_n}$ appears, and takes the place of $\alpha^{(i)}_{j_1 \cdots j_n}$. Thus all of the $b^{(i)}_{j_1 \cdots j_n}$ and all of the $\alpha^{(i)}_{j_1 \cdots j_n}$ are determined with the exception of those $\alpha^{(i)}_{j_1 \cdots j_n}$ for which $\sum_{k=1}^{n} j_k \sigma_k - \sigma_i = 0$, and these are left arbitrary. Let us denote these arbitraries by $A^{(i)}_{j_1 \cdots j_n}$ and suppose that $|A^{(i)}_{j_1 \cdots j_n}| < M$. Let us suppose further that the computation has been carried out so far that all of the $b^{(i)}_{j_1 \cdots j_n}$ have been determined. Then all of the terms in the substitution $x_i = y_i + \cdots$ are known up to, but not including, terms of degree s. We write therefore

$$(36) \qquad x_i = y_i + f_i(y_1, \cdots, y_n) + R_i(y_1, \cdots, y_n),$$

where the f_i are known polynomials of degree $s - 1$ and order two, and the R_i are power series of order s.

It is seen from (35) that the functions R_i are determined by the equations

$$(37) \qquad \sum_{k=1}^{n} (\sigma_k y_k + p_k) \frac{\partial R_i}{\partial y_k} - \sigma_i R_i = S_i(y_1, \cdots, y_n; R_1, \cdots, R_n),$$

where the S_i are power series of order s in the arguments indicated. Since the S_i are convergent power series of order s it is no restriction upon the generality to suppose that the moduli of the coefficients are all less than $\nu_i > 0$. If the positive constant a is properly chosen the solutions of equations (37) are dominated by the solutions of the equations

$$(38) \quad \epsilon \left[\sum_{j=1}^{n} \left\{ y_j - \frac{a}{n}(y^2 + \cdots + y^{s-1}) \right\} \frac{\partial T_i}{\partial y_j} - T_i \right] = \frac{\epsilon}{n} \frac{(y + T)^s}{1 - (y + T)} \qquad (i = 1, \cdots, n),$$

where for brevity we have taken $y = \sum_{k=1}^{n} y_k$, $T = \sum_{k=1}^{n} T_k$.

The solution of the system of equations (38) reduces, on taking the sum of all of the equations, to the single equation

$$(39) \qquad [y - a(y^2 + \cdots + y^{s-1})] \frac{dT}{dy} - T = \frac{(y + T)^s}{1 - (y + T)}.$$

On substituting $T = v - y$, or $y + T = v$, we get

$$(40) \quad [y - a(y^2 + \cdots + y^{s-1})] \frac{dv}{dy} - v = \frac{v^s}{1 - v} - a(y^2 + \cdots + y^{s-1}).$$

The solution of equation (40) is dominated by the solution of

$$[y - a(y^2 + \cdots + y^{s-1})]\frac{dv}{dy} - v = \frac{v^s}{1 - v}. \tag{41}$$

On integrating this equation and choosing the constant of integration so that $v = y + \cdots$, we get

$$\log\frac{v}{y} - \frac{1}{s-1}v^{s-1} - \cdots = a(y + \tfrac{1}{2}y^2 + \cdots + \frac{1}{s-2}y^{s-2}) + \cdots. \tag{42}$$

If now we set $v = (1 + w)y$ it is seen that (42) admits a unique solution for w as a power series in y vanishing with y. Since this series is convergent and dominates the solutions of (37), it follows that the solutions of (37) are convergent. We have then

THEOREM II. *If, in the system of differential equations*

$$\frac{dx_i}{dt} = \sigma_i x_i + X_i(x_1, \cdots, x_n) \qquad (i = 1, \cdots, n),$$

where the X_i are convergent power series in $x_1, \cdots, x_n$ of order not less than two, the σ_i, considered as points in the complex plane, lie all on one side of a straight line passing through the origin, then there exists a convergent substitution

$$x_i = y_i + P_i(y_1, \cdots, y_n) \qquad (i = 1, \cdots, n),$$

where the P_i are power series in $y_1, \cdots, y_n$ of order not less than two, such that the variables y_i satisfy the differential equations

$$\frac{dy_i}{dt} = \sigma_i y_i + \sum{}^* b^{(i)}_{j_1 \cdots j_n} y_1^{j_1} \cdots y_n^{j_n} \qquad \left(\sum_{k=1}^{n} j_k \geqq 2\right), \tag{43}$$

where the terms included under the symbol Σ^* include all terms for which

$$j_1\sigma_1 + j_2\sigma_2 + \cdots + j_n\sigma_n - \sigma_i = 0,$$

and where the $b^{(i)}_{j_1 \cdots j_n}$ are properly chosen constants.

Notwithstanding the fact that equations (43) are non-linear they are easily integrated as is illustrated by the following example:

Let us suppose that $\sigma_1 = 1$, $\sigma_2 = 2$, and $\sigma_3 = 3$. Then the following relations exist

$$2\sigma_1 - \sigma_2 = 0, \qquad \sigma_1 + \sigma_2 - \sigma_3 = 0, \qquad 3\sigma_1 - \sigma_3 = 0,$$

and there are no others. Equations (43) become

$$\begin{aligned} y_1' &= y_1, \\ y_2' &= 2y_2 + Ay_1^2, \\ y_3' &= 3y_3 + By_1 y_2 + Cy_1^3, \end{aligned} \tag{44}$$

where A, B, and C are certain constants. The solutions of equations (44) are readily found to be

$$y_1 = c_1 e^t,$$

$$y_2 = [c_2 + c_1^2 At] e^{2t},$$

$$y_3 = [c_3 + (c_1 c_2 B + c_1 C)t + \tfrac{1}{2}c_1^3 ABt^2] e^{3t}.$$

The solutions of these equations are analogous to the solutions of non-homogeneous linear differential equations in which the non-homogeneous terms have the same period as the homogeneous terms.

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